

# ON THE DIFFERENCE OF SPECTRAL PROJECTIONS

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**ABSTRACT.** For a semibounded self-adjoint operator  $T$  and a compact self-adjoint operator  $S$  acting on a complex separable Hilbert space of infinite dimension, we study the difference  $D(\lambda) := E_{(-\infty, \lambda)}(T+S) - E_{(-\infty, \lambda)}(T)$ ,  $\lambda \in \mathbb{R}$ , of the spectral projections associated with the open interval  $(-\infty, \lambda)$ .

In the case when  $S$  is of rank one, we show that  $D(\lambda)$  is unitarily equivalent to a block diagonal operator  $\Gamma_\lambda \oplus 0$ , where  $\Gamma_\lambda$  is a bounded self-adjoint Hankel operator, for all  $\lambda \in \mathbb{R}$  except for at most countably many  $\lambda$ .

If, more generally,  $S$  is compact, then we obtain that  $D(\lambda)$  is unitarily equivalent to an essentially Hankel operator (in the sense of Martínez-Avendaño) on  $\ell^2(\mathbb{N}_0)$  for all  $\lambda \in \mathbb{R}$  except for at most countably many  $\lambda$ .

## 1. INTRODUCTION AND MAIN RESULTS

When a self-adjoint operator  $T$  is perturbed by a bounded self-adjoint operator  $S$ , it is important to investigate the (spectral) properties of the difference

$$f(T+S) - f(T),$$

where  $f$  is a real-valued Borel function on  $\mathbb{R}$ . It is also of interest to predict the smoothness of the mapping  $S \mapsto f(T+S) - f(T)$  with respect to the smoothness of  $f$ . There is a vast amount of literature dedicated to these problems, see, e. g., Kreĭn, Farforovskaja, Peller, Birman, Solomyak, Pushnitski, Yafaev [4, 9, 16, 17, 24, 25, 27–30], and the references therein.

It is well known (see Kreĭn [16]; see also Peller [25]) that if  $f$  is an infinitely differentiable function with compact support and  $S$  is trace class, then  $f(T+S) - f(T)$  is a trace class operator.

On the other hand, if  $f = \mathbb{1}_{(-\infty, \lambda)}$  is the characteristic function of the interval  $(-\infty, \lambda)$  with  $\lambda$  in the essential spectrum of  $T$ , then it may occur that

$$f(T+S) - f(T)$$

is not compact, see Kreĭn's example [15, 16]. In the latter example,  $S$  is a rank one operator, and the difference  $\mathbb{1}_{(-\infty, \lambda)}(T+S) - \mathbb{1}_{(-\infty, \lambda)}(T)$  is a bounded self-adjoint Hankel integral operator on  $L^2(0, \infty)$  that can be computed explicitly for all  $0 < \lambda < 1$ .

Formally, a bounded Hankel integral operator  $\Gamma$  on  $L^2(0, \infty)$  is a bounded integral operator such that the kernel function  $k$  of  $\Gamma$  depends only on the sum of the variables:

$$(\Gamma g)(x) = \int_0^\infty k(x+y)g(y)dy, \quad g \in L^2(0, \infty).$$

For an introduction to the theory of Hankel operators, we refer to Peller's book [26].

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Inspired by Kreĭn's example, we may pose the following question.

**Question 1.** *Let  $\lambda \in \mathbb{R}$ . Is it true that*

$$D(\lambda) = E_{(-\infty, \lambda)}(T + S) - E_{(-\infty, \lambda)}(T),$$

*the difference of the spectral projections, is unitarily equivalent to a bounded self-adjoint Hankel operator, provided that  $T$  is semibounded and  $S$  is of rank one?*

Pushnitski [27–30] and Yafaev [30] have been studying the spectral properties of the operator  $D(\lambda)$  in connection with scattering theory. If the absolutely continuous spectrum of  $T$  contains an open interval and under some smoothness assumptions, the results of Pushnitski and Yafaev are applicable, cf. Section 5 below. In this case, the essential spectrum of  $D(\lambda)$  is a symmetric interval around zero.

Here and for the rest of this paper, we consider a semibounded self-adjoint operator  $T$  acting on a complex separable Hilbert space  $\mathfrak{H}$  of infinite dimension. We denote the spectrum and the essential spectrum of  $T$  by  $\sigma(T)$  and  $\sigma_{\text{ess}}(T)$ , respectively.

Furthermore, we denote by  $\text{span}\{x_i \in \mathfrak{H} : i \in \mathcal{J}\}$  the linear span generated by the vectors  $x_i$ ,  $i \in \mathcal{J}$ , where  $\mathcal{J}$  is some index set. If there exists a vector  $x \in \mathfrak{H}$  such that

$$\overline{\text{span}\{E_{\Omega}(T)x : \Omega \in \mathcal{B}(\mathbb{R})\}} := \overline{\text{span}\{E_{\Omega}(T)x : \Omega \in \mathcal{B}(\mathbb{R})\}} = \mathfrak{H},$$

then  $x$  is called *cyclic* for  $T$ . Here  $\mathcal{B}(\mathbb{R})$  denotes the sigma-algebra of Borel sets of  $\mathbb{R}$ .

The following theorem is the main result of this paper.

**Theorem 2.** *Let  $T$  and  $S$  be a semibounded self-adjoint operator and a self-adjoint operator of rank one acting on  $\mathfrak{H}$ , respectively. Then there exists a number  $k$  in  $\mathbb{N} \cup \{0\}$  such that for all  $\lambda$  in  $\mathbb{R}$  except for at most countably many  $\lambda$  in  $\sigma_{\text{ess}}(T)$ , the operator  $D(\lambda)$  is unitarily equivalent to a block diagonal operator  $\Gamma_{\lambda} \oplus 0$  on  $L^2(0, \infty) \oplus \mathbb{C}^k$ , where  $\Gamma_{\lambda}$  is a bounded self-adjoint Hankel integral operator on  $L^2(0, \infty)$ .*

The theory of bounded self-adjoint Hankel operators has been studied intensively by Rosenblum, Howland, Megretskiĭ, Peller, Treil, and others, see [11, 22, 31, 32]. In their 1995 paper [22], Megretskiĭ, Peller, and Treil have shown that every bounded self-adjoint Hankel operator can be characterized by three properties concerning the spectrum and the multiplicity in the spectrum, see [22, Theorem 1].

We present a version of [22, Theorem 1] for differences of two orthogonal projections in Section 2, see Theorem 2.2 below.

Denote by  $\ell^2(\mathbb{N}_0)$  the space of all complex square summable one-sided sequences  $x = (x_0, x_1, \dots)$ . A bounded operator  $H$  on  $\ell^2(\mathbb{N}_0)$  is called *essentially Hankel* if  $A^*H - HA$  is compact, where  $A : (x_0, x_1, \dots) \mapsto (0, x_0, x_1, \dots)$  denotes the forward shift on  $\ell^2(\mathbb{N}_0)$ . The set of essentially Hankel operators was introduced in [20] by Martínez-Avendaño. Clearly, every operator of the form 'Hankel plus compact' is essentially Hankel, but the converse is not true (see [20, Theorem 3.8]).

For compact perturbations  $S$ , we will prove the following version of Theorem 2.

**Theorem 3.** *Let  $T$  and  $S$  be a semibounded self-adjoint operator and a compact self-adjoint operator acting on  $\mathfrak{H}$ , respectively. Let  $1/4 > a_1 > a_2 > \dots > 0$  be an arbitrary decreasing null sequence of real numbers. Then for all  $\lambda$  in  $\mathbb{R}$  except for at most countably many  $\lambda$  in  $\sigma_{\text{ess}}(T)$ , there exist a bounded self-adjoint Hankel operator  $\Gamma_\lambda$  and a compact self-adjoint operator  $K_\lambda$  acting on  $\ell^2(\mathbb{N}_0)$  with the following properties:*

- (1)  $D(\lambda)$  is unitarily equivalent to  $\Gamma_\lambda + K_\lambda$ .
- (2) either  $K_\lambda$  is a finite rank operator or  $\nu_j(\lambda)/a_j \rightarrow 0$  as  $j \rightarrow \infty$ , where  $\nu_1(\lambda), \nu_2(\lambda), \dots$  denote the nonzero eigenvalues of  $K_\lambda$  ordered by decreasing modulus (with multiplicity taken into account).

In particular,  $\Gamma_\lambda + K_\lambda$  is essentially Hankel.

Moreover, the operator  $K_\lambda$  in Theorem 3 can always be chosen of finite rank if  $S$  is of finite rank.

In Sections 3–5, the operator  $T$  is supposed to be bounded. The case when  $T$  is semibounded (but not bounded) will be reduced to the bounded case by means of resolvents, see Subsection 6.2 and the remark in Subsection 6.1.

In Section 3, we will show that the dimensions of  $\text{Ker}(D(\lambda) \pm I)$  differ by at most  $N \in \mathbb{N}$  if  $S$  is of rank  $N$ , where  $I$  denotes the identity operator. We write this as

$$(1.1) \quad |\dim \text{Ker}(D(\lambda) - I) - \dim \text{Ker}(D(\lambda) + I)| \leq \text{rank } S, \quad \lambda \in \mathbb{R}.$$

Furthermore, an example is given where equality is attained.

However, there may exist  $\lambda \in \mathbb{R}$  such that

$$\dim \text{Ker}(D(\lambda) - I) = \infty \quad \text{and} \quad \text{Ker}(D(\lambda) + I) = \{0\}$$

if  $S$  is a compact operator with infinite dimensional range.

Section 4 provides a list of sufficient conditions so that Question 1 has a positive answer, see Proposition 4.1.

Moreover, if  $S = \langle \cdot, \varphi \rangle \varphi$  is a rank one operator and the vector  $\varphi$  is cyclic for  $T$ , then we will show in Theorem 4.3 that the kernel of  $D(\lambda)$  is trivial for all  $\lambda$  in the interval  $(\min \sigma_{\text{ess}}(T), \max \sigma_{\text{ess}}(T))$  and infinite dimensional for all  $\lambda$  in  $\mathbb{R} \setminus [\min \sigma_{\text{ess}}(T), \max \sigma_{\text{ess}}(T)]$ .

In the case when  $\varphi$  is not cyclic for  $T$ , Example 4.2 shows that Question 1 may have to be answered negatively. In this situation, we need to consider the block operator representation of  $D(\lambda)$  with respect to the orthogonal subspaces  $\overline{\text{span}}\{T^j \varphi : j \in \mathbb{N}_0\}$  and  $\mathfrak{H} \ominus \overline{\text{span}}\{T^j \varphi : j \in \mathbb{N}_0\}$  of  $\mathfrak{H}$ , see Subsection 4.2.

In Section 5, we will show that the operator  $D(\lambda)$  is non-invertible for all  $\lambda$  in  $\mathbb{R}$  except for at most countably many  $\lambda$  in  $\sigma_{\text{ess}}(T)$ , see Theorem 5.1.

Section 6 completes the proofs of Theorems 2 and 3. In particular, it is shown that  $D(\lambda)$  is unitarily equivalent to a self-adjoint Hankel operator of finite rank for all  $\lambda \in \mathbb{R}$  if  $T$  has a purely discrete spectrum and  $S$  is a rank one operator (see Proposition 6.6 and p. 19).

Some examples, including the almost Mathieu operator, are discussed in Section 7 below.

The results of this paper will be part of the author's Ph.D. thesis at Johannes Gutenberg University Mainz.

## 2. THE MAIN TOOLS

In this section, we present the main tools for the proofs of Theorems 2 and 3. First, we state a lemma which follows immediately from [7, Theorem 6.1].

**Lemma 2.1.** *Let  $\Gamma$  be the difference of two orthogonal projections. Then  $\sigma(\Gamma) \subset [-1, 1]$ . Moreover, the restricted operators  $\Gamma|_{\mathfrak{H}_0}$  and  $(-\Gamma)|_{\mathfrak{H}_0}$  are unitarily equivalent, where the closed subspace  $\mathfrak{H}_0 := [\text{Ker}(\Gamma - I) \oplus \text{Ker}(\Gamma + I)]^\perp$  of  $\mathfrak{H}$  is reducing for  $\Gamma$ .*

In [22], Megretskiĭ, Peller, and Treil solved the inverse spectral problem for self-adjoint Hankel operators. In our situation, [22, Theorem 1] reads as follows:

**Theorem 2.2.** *The difference  $\Gamma$  of two orthogonal projections is unitarily equivalent to a bounded self-adjoint Hankel operator if and only if the following three conditions hold:*

- (C1) *either  $\text{Ker } \Gamma = \{0\}$  or  $\dim \text{Ker } \Gamma = \infty$ ;*
- (C2)  *$\Gamma$  is non-invertible;*
- (C3)  *$|\dim \text{Ker}(\Gamma - I) - \dim \text{Ker}(\Gamma + I)| \leq 1$ .*

If  $\dim \text{Ker}(\Gamma - I) = \infty$  or  $\dim \text{Ker}(\Gamma + I) = \infty$ , then (C3) has to be understood as  $\dim \text{Ker}(\Gamma - I) = \dim \text{Ker}(\Gamma + I) = \infty$  (cf. [22, p. 249]).

*Proof of Theorem 2.2.* Combine Lemma 2.1 and [22, Theorem 1]. □

As will be shown in Section 3, the operator  $D(\lambda)$  satisfies condition (C3) for all  $\lambda \in \mathbb{R}$  if  $S$  is a rank one operator. Therefore, a sufficient condition for  $D(\lambda)$  to be unitarily equivalent to a bounded self-adjoint Hankel operator is given by:

*the kernel of  $D(\lambda)$  is infinite dimensional.*

In Proposition 4.1 below, we present a list of sufficient conditions such that the kernel of  $D(\lambda)$  is infinite dimensional.

More generally, a self-adjoint *block-Hankel operator of order  $N$*  is a block-Hankel matrix  $(a_{j+k})_{j,k \in \mathbb{N}_0}$ , where  $a_j$  is an  $N \times N$  matrix for every  $j$ , see [22, p. 247]. We will need the following version of Theorem 2.2:

**Theorem 2.3.** *The difference  $\Gamma$  of two orthogonal projections is unitarily equivalent to a bounded self-adjoint block-Hankel operator of order  $N$  if and only if the following three conditions hold:*

- (C1) *either  $\text{Ker } \Gamma = \{0\}$  or  $\dim \text{Ker } \Gamma = \infty$ ;*
- (C2)  *$\Gamma$  is non-invertible;*
- (C3) <sub>$N$</sub>   *$|\dim \text{Ker}(\Gamma - I) - \dim \text{Ker}(\Gamma + I)| \leq N$ .*

Again, if  $\dim \text{Ker}(\Gamma - I) = \infty$  or  $\dim \text{Ker}(\Gamma + I) = \infty$ , then (C3) <sub>$N$</sub>  has to be understood as  $\dim \text{Ker}(\Gamma - I) = \dim \text{Ker}(\Gamma + I) = \infty$ .

*Proof of Theorem 2.3.* Combine Lemma 2.1 and [22, Theorem 2]. □

## 3. ON THE DIMENSION OF $\text{Ker}(D(\lambda) \pm I)$

In this section, the self-adjoint operator  $T$  is assumed to be bounded. The main purpose of this section is to show that the dimensions of  $\text{Ker}(D(\lambda) \pm I)$  do not exceed the rank of the perturbation  $S$ , see Lemma 3.1 below.

In particular, condition (C3) <sub>$N$</sub>  in Theorem 2.3 is fulfilled for all  $\lambda \in \mathbb{R}$  if the rank of  $S$  is equal to  $N \in \mathbb{N}$ .

**Lemma 3.1.** *Let  $T$  and  $S$  be a bounded self-adjoint operator and a self-adjoint operator of finite rank  $N$  acting on  $\mathfrak{H}$ , respectively. Then for all  $\lambda$  in  $\mathbb{R}$ , one has*

$$\dim \operatorname{Ker}(D(\lambda) \pm I) \leq N.$$

*Proof.* Let us write  $P_\lambda := E_{(-\infty, \lambda)}(T + S)$  and  $Q_\lambda := E_{(-\infty, \lambda)}(T)$ .

We will only show that  $\dim \operatorname{Ker}(P_\lambda - Q_\lambda - I) \leq N$ ; the other inequality is proved analogously.

Assume for contradiction that there exists an orthonormal system  $x_1, \dots, x_{N+1}$  in  $\operatorname{Ker}(P_\lambda - Q_\lambda - I)$ . Choose a normed vector  $\tilde{x}$  in

$$\operatorname{span}\{x_1, \dots, x_{N+1}\} \cap (\operatorname{Ran} S)^\perp \neq \{0\}.$$

Hence  $P_\lambda \tilde{x} = \tilde{x}$  and  $Q_\lambda \tilde{x} = 0$  and this implies

$$\langle (T + S)\tilde{x}, \tilde{x} \rangle < \lambda \quad \text{and} \quad \langle T\tilde{x}, \tilde{x} \rangle \geq \lambda$$

so that

$$\lambda > \langle (T + S)\tilde{x}, \tilde{x} \rangle = \langle T\tilde{x}, \tilde{x} \rangle \geq \lambda,$$

which is a contradiction.  $\square$

**Remark.** If we consider an unbounded self-adjoint operator  $T$ , then the proof of Lemma 3.1 does not work, because  $\tilde{x}$  might not belong to the domain of  $T$ .

The following example shows that Inequality (1.1) above is optimal.

**Example 3.2.** (1) Consider the bounded self-adjoint diagonal operator

$$T = \operatorname{diag}(-1, -1/2, -1/3, -1/4, \dots) : \ell^2(\mathbb{N}_0) \rightarrow \ell^2(\mathbb{N}_0)$$

and, for  $N \in \mathbb{N}$ , the self-adjoint diagonal operator

$$S = \operatorname{diag}(\underbrace{-1, \dots, -1}_{N \text{ times}}, 0, \dots) : \ell^2(\mathbb{N}_0) \rightarrow \ell^2(\mathbb{N}_0).$$

Then  $S$  is of rank  $N$ , and we see that

$$\dim \operatorname{Ker}(E_{(-\infty, \lambda)}(T + S) - E_{(-\infty, \lambda)}(T) - I) = N$$

$$\text{and } \operatorname{Ker}(E_{(-\infty, \lambda)}(T + S) - E_{(-\infty, \lambda)}(T) + I) = \{0\}$$

for all  $\lambda \in (-1 - 1/N, -1)$ .

(2) Let  $a_0 := -1$ ,  $a_1 := -1/2$ ,  $a_2 := -1/3$ . Consider the bounded self-adjoint diagonal operator

$$T = \operatorname{diag}\left(a_0, a_0 + \frac{1/2}{4}, a_1, a_1 + \frac{1/6}{4}, a_0 + \frac{1/2}{5}, a_1 + \frac{1/6}{5}, a_0 + \frac{1/2}{6}, \dots\right)$$

on  $\ell^2(\mathbb{N}_0)$ . Since  $|a_0 - a_1| = 1/2$  and  $|a_1 - a_2| = 1/6$ , it follows that the compact self-adjoint diagonal operator

$$S = -2 \cdot \operatorname{diag}\left(0, \frac{1/2}{4}, 0, \frac{1/6}{4}, \frac{1/2}{5}, \frac{1/6}{5}, \frac{1/2}{6}, \dots\right) : \ell^2(\mathbb{N}_0) \rightarrow \ell^2(\mathbb{N}_0)$$

is such that

$$(+)\left\{\begin{array}{l} \dim \operatorname{Ker}(E_{(-\infty, \lambda)}(T + S) - E_{(-\infty, \lambda)}(T) - I) = \infty \\ \text{and } \operatorname{Ker}(E_{(-\infty, \lambda)}(T + S) - E_{(-\infty, \lambda)}(T) + I) = \{0\} \end{array}\right.$$

for  $\lambda \in \{-1, -1/2\}$ .

Clearly, this example can be extended such that (+) holds for all  $\lambda$  in  $\{-1, -1/2, -1/3, \dots\}$ .

#### 4. ON THE DIMENSION OF $\text{Ker } D(\lambda)$

In this section, we deal with the question whether the operator  $D(\lambda)$  fulfills condition (C1) in Theorem 2.3.

Suppose that the self-adjoint operators  $T$  and  $S$  are bounded and of finite rank, respectively. We will provide a list of sufficient conditions such that the kernel of  $D(\lambda)$  is infinite dimensional for all  $\lambda$  in  $\mathbb{R}$ .

Furthermore, we will prove that the kernel of  $D(\lambda)$  is trivial for all  $\lambda$  in the interval  $(\min \sigma_{\text{ess}}(T), \max \sigma_{\text{ess}}(T))$  and infinite dimensional for all  $\lambda$  in  $\mathbb{R} \setminus [\min \sigma_{\text{ess}}(T), \max \sigma_{\text{ess}}(T)]$ , provided that  $S = \langle \cdot, \varphi \rangle \varphi$  is a rank one operator and the vector  $\varphi$  is cyclic for  $T$ , see Theorem 4.3 below.

**4.1. Sufficient conditions such that  $\dim \text{Ker } D(\lambda) = \infty$ .** Let  $\lambda \in \mathbb{R}$ . If the kernel of  $D(\lambda) = E_{(-\infty, \lambda)}(T + S) - E_{(-\infty, \lambda)}(T)$  is infinite dimensional, then  $D(\lambda)$  fulfills conditions (C1) and (C2) in Theorem 2.3.

Let  $N \in \mathbb{N}$  be the rank of  $S$ . The following proposition provides a list of sufficient conditions such that the kernel of  $D(\lambda)$  is infinite dimensional.

**Proposition 4.1.** *If at least one of the following three cases occurs for  $X = T$  or for  $X = T + S$ , then the operator  $D(\lambda)$  is unitarily equivalent to a bounded self-adjoint block-Hankel operator of order  $N$  with infinite dimensional kernel for all  $\lambda \in \mathbb{R}$ .*

- (1) *The spectrum of  $X$  contains an eigenvalue of infinite multiplicity. In particular, this pertains to the case when the range of  $X$  is finite dimensional.*
- (2) *The spectrum of  $X$  contains infinitely many eigenvalues with multiplicity at least  $N + 1$ .*
- (3) *The spectrum of the restricted operator  $X|_{\mathfrak{E}^\perp}$  has multiplicity at least  $N + 1$  (not necessarily uniform), where  $\mathfrak{E} := \{x \in \mathfrak{H} : x \text{ is an eigenvector of } X\}$ .*

*Proof.* By Lemma 3.1, we know that condition (C3)<sub>N</sub> in Theorem 2.3 holds true for all  $\lambda \in \mathbb{R}$ . It remains to show that  $\dim \text{Ker } D(\lambda) = \infty$  for all  $\lambda \in \mathbb{R}$ .

First, suppose that there exists an eigenvalue  $\lambda_0$  of  $X = T$  with multiplicity  $m \geq N + 1$ , i. e.  $m \in \{N + 1, N + 2, \dots\} \cup \{\infty\}$ . Define

$$\mathfrak{M} := (\text{Ran } E_{\{\lambda_0\}}(T)) \cap (\text{Ran } S)^\perp \neq \{0\}.$$

It is easy to show that  $\mathfrak{M}$  is a closed subspace of  $\mathfrak{H}$  such that

- $\dim \mathfrak{M} \geq m - N$ ,
- $T|_{\mathfrak{M}} = (T + S)|_{\mathfrak{M}}$ ,
- $T(\mathfrak{M}) \subset \mathfrak{M}$  and  $T(\mathfrak{M}^\perp) \subset \mathfrak{M}^\perp$ ,
- $(T + S)(\mathfrak{M}) \subset \mathfrak{M}$  and  $(T + S)(\mathfrak{M}^\perp) \subset \mathfrak{M}^\perp$ .

Therefore,  $\mathfrak{M}$  is contained in the kernel of  $D(\lambda)$  for all  $\lambda \in \mathbb{R}$ .

It follows that the kernel of  $D(\lambda)$  is infinite dimensional for all  $\lambda \in \mathbb{R}$  whenever case (1) or case (2) occur for the operator  $X = T$ ; in the case when  $X = T + S$  the proof runs analogously.

Now suppose that case (3) occurs for  $X = T$ . Write

$$S = \sum_{k=1}^N \alpha_k \langle \cdot, \varphi_k \rangle \varphi_k : \mathfrak{H} \rightarrow \mathfrak{H},$$

where  $\varphi_1, \dots, \varphi_N$  form an orthonormal system in  $\mathfrak{H}$  and  $\alpha_1, \dots, \alpha_N$  are nonzero real numbers. Define the closed subspace  $\mathfrak{N} := \overline{\text{span}} \{T^j \varphi_k : j \in \mathbb{N}_0, k = 1, \dots, N\}$  of  $\mathfrak{H}$ . It is well known that

- $T|_{\mathfrak{N}^\perp} = (T + S)|_{\mathfrak{N}^\perp}$ ,
- $T(\mathfrak{N}) \subset \mathfrak{N}$  and  $T(\mathfrak{N}^\perp) \subset \mathfrak{N}^\perp$ ,
- $(T + S)(\mathfrak{N}) \subset \mathfrak{N}$  and  $(T + S)(\mathfrak{N}^\perp) \subset \mathfrak{N}^\perp$ .

Therefore,  $\mathfrak{N}^\perp$  is contained in the kernel of  $D(\lambda)$  for all  $\lambda \in \mathbb{R}$ . A standard proof using the theory of direct integrals (see [5, Chapter 7], see in particular [5, Theorem 1, p. 177]) shows that  $\mathfrak{N}^\perp$  is infinite dimensional.

If  $X = T + S$ , then the proof runs analogously.

Now the proof is complete.  $\square$

**4.2. The case when  $S$  is a rank one operator.** For the rest of this section, let us assume that  $S = \langle \cdot, \varphi \rangle \varphi$  is a rank one operator.

The following example illustrates that  $\dim \text{Ker } D(\lambda)$  may attain every value in  $\mathbb{N}$ , provided that  $\varphi$  is not cyclic for  $T$ . Recall that when  $\dim \text{Ker } D(\lambda)$  is neither zero nor infinity, Theorem 2.2 shows that Question 1 has to be answered negatively.

**Example 4.2.** *Essentially, this is an application of Kreĭn's example from [16, pp. 622–624].*

*Let  $0 < \lambda < 1$ . Consider the bounded self-adjoint integral operators  $A_j$ ,  $j = 0, 1$ , with kernel functions*

$$a_0(x, y) = \begin{cases} \sinh(x)e^{-y} & \text{if } x \leq y \\ \sinh(y)e^{-x} & \text{if } x \geq y \end{cases} \quad \text{and} \quad a_1(x, y) = \begin{cases} \cosh(x)e^{-y} & \text{if } x \leq y \\ \cosh(y)e^{-x} & \text{if } x \geq y \end{cases}$$

*on the Hilbert space  $L^2(0, \infty)$ . By [16, pp. 622–624], we know that  $A_0 - A_1$  is of rank one and that the difference  $E_{(-\infty, \lambda)}(A_0) - E_{(-\infty, \lambda)}(A_1)$  is a Hankel operator. Furthermore, it was shown in [15, Theorem 1] that  $E_{(-\infty, \lambda)}(A_0) - E_{(-\infty, \lambda)}(A_1)$  has a simple purely absolutely continuous spectrum filling in the interval  $[-1, 1]$ . In particular, the kernel of*

$$E_{(-\infty, \lambda)}(A_0) - E_{(-\infty, \lambda)}(A_1)$$

*is trivial. Let  $k \in \mathbb{N}$ . Now consider block diagonal operators*

$$\tilde{A}_j := A_j \oplus M : L^2(0, \infty) \oplus \mathbb{C}^k \rightarrow L^2(0, \infty) \oplus \mathbb{C}^k, \quad j = 0, 1,$$

*where  $M \in \mathbb{C}^{k \times k}$  is an arbitrary fixed self-adjoint matrix. Then one has*

$$\dim \text{Ker} \left( E_{(-\infty, \lambda)}(\tilde{A}_0) - E_{(-\infty, \lambda)}(\tilde{A}_1) \right) = k.$$

The following consideration shows that (up to at most countably many  $\lambda$  in the essential spectrum of  $T$ ) this is the only type of counterexample to Question 1 above.

The closed subspace  $\mathfrak{N}^\perp$  of  $\mathfrak{H}$  might be trivial, finite dimensional, or infinite dimensional, where  $\mathfrak{N} := \overline{\text{span}}\{T^j \varphi : j \in \mathbb{N}_0\}$ .

**Case 1.** If  $\mathfrak{N}^\perp$  is trivial, then  $\varphi$  is cyclic for  $T$  and Proposition 6.1 below implies that  $D(\lambda)$  is unitarily equivalent to a bounded self-adjoint Hankel operator for all  $\lambda$  in  $\mathbb{R}$  except for at most countably many  $\lambda$  in  $\sigma_{\text{ess}}(T)$ .



**Case 2.** Suppose that  $\dim(\mathfrak{N}^\perp) =: k \in \mathbb{N}$ . We will reduce this situation to the first case. Let us identify  $\mathfrak{N}^\perp$  with  $\mathbb{C}^k$ . The restricted operators  $T|_{\mathfrak{N}^\perp}$  and  $(T + S)|_{\mathfrak{N}^\perp}$  coincide on  $\mathfrak{N}^\perp$ , and since  $\mathfrak{N}$  reduces both  $T$  and  $T + S$  there exists a self-adjoint matrix  $M$  in  $\mathbb{C}^{k \times k}$  such that  $T$  and  $T + S$  can be identified with the block diagonal operators  $T|_{\mathfrak{N}} \oplus M$  and  $(T + S)|_{\mathfrak{N}} \oplus M$  acting on  $\mathfrak{N} \oplus \mathbb{C}^k$ , respectively. Therefore,

$$D(\lambda) = \left( E_{(-\infty, \lambda)}(T|_{\mathfrak{N}} + S|_{\mathfrak{N}}) - E_{(-\infty, \lambda)}(T|_{\mathfrak{N}}) \right) \oplus 0, \quad \lambda \in \mathbb{R},$$

and  $\varphi$  is cyclic for  $T|_{\mathfrak{N}}$ .

**Case 3.** Since  $\mathfrak{N}^\perp \subset \text{Ker } D(\lambda)$  for all  $\lambda$  in  $\mathbb{R}$ , it follows from Lemma 3.1 and Theorem 2.2 that  $D(\lambda)$  is unitarily equivalent to a bounded self-adjoint Hankel operator for all  $\lambda$  in  $\mathbb{R}$  if  $\mathfrak{N}^\perp$  is infinite dimensional.

**4.3. The case when  $\varphi$  is cyclic for  $T$ .** This subsection is devoted to the proof of the following theorem:

**Theorem 4.3.** *Suppose that the self-adjoint operators  $T$  and  $S = \langle \cdot, \varphi \rangle \varphi$  are bounded and of rank one, respectively, and that the vector  $\varphi$  is cyclic for  $T$ . Let  $\lambda \in \mathbb{R} \setminus \{\min \sigma_{\text{ess}}(T), \max \sigma_{\text{ess}}(T)\}$ . Then the kernel of  $D(\lambda)$  is*

- (1) *infinite dimensional if and only if  $\lambda \in \mathbb{R} \setminus [\min \sigma_{\text{ess}}(T), \max \sigma_{\text{ess}}(T)]$ .*
- (2) *trivial if and only if  $\lambda \in (\min \sigma_{\text{ess}}(T), \max \sigma_{\text{ess}}(T))$ .*

*In particular, one has*

$$\text{either } \text{Ker } D(\lambda) = \{0\} \quad \text{or} \quad \dim \text{Ker } D(\lambda) = \infty.$$

The proof is based on a result by Liaw and Treil [19] and some harmonic analysis.

Theorem 4.3 will be an important ingredient in the proof of Proposition 6.1 below. Likewise, it is of independent interest. Note that, according to Theorem 4.3, the kernel of  $D(\lambda)$  is trivial for *all*  $\lambda$  between  $\min \sigma_{\text{ess}}(T)$  and  $\max \sigma_{\text{ess}}(T)$ , no matter if the interval  $(\min \sigma_{\text{ess}}(T), \max \sigma_{\text{ess}}(T))$  contains points from the resolvent set of  $T$ , isolated eigenvalues of  $T$ , etc.

It will be useful to write  $S = S_\alpha = \alpha \langle \cdot, \varphi \rangle \varphi$  for some real number  $\alpha \neq 0$  such that  $\|\varphi\| = 1$ .

Let  $\lambda \in \mathbb{R}$ . Again, we write

$$P_\lambda = E_{(-\infty, \lambda)}(T + S_\alpha) \quad \text{and} \quad Q_\lambda = E_{(-\infty, \lambda)}(T).$$

Observe that the kernel of  $P_\lambda - Q_\lambda$  is equal to the orthogonal sum of  $(\text{Ran } P_\lambda) \cap (\text{Ran } Q_\lambda)$  and  $(\text{Ker } P_\lambda) \cap (\text{Ker } Q_\lambda)$ . Therefore, we will investigate the dimensions of  $(\text{Ran } P_\lambda) \cap (\text{Ran } Q_\lambda)$  and  $(\text{Ker } P_\lambda) \cap (\text{Ker } Q_\lambda)$  separately.

Now we follow [19, pp. 1948–1949] in order to represent the operators  $T$  and  $T + S_\alpha$  such that [19, Theorem 2.1] is applicable.

Define Borel probability measures  $\mathbb{P}$  and  $\mathbb{P}_\alpha$  on  $\mathbb{R}$  by

$$\mathbb{P}(\Omega) := \langle E_\Omega(T)\varphi, \varphi \rangle \quad \text{and} \quad \mathbb{P}_\alpha(\Omega) := \langle E_\Omega(T + S_\alpha)\varphi, \varphi \rangle, \quad \Omega \in \mathcal{B}(\mathbb{R}),$$

respectively. According to [33, Proposition 5.18], there exist unitary operators  $U : \mathfrak{H} \rightarrow L^2(\mathbb{P})$  and  $U_\alpha : \mathfrak{H} \rightarrow L^2(\mathbb{P}_\alpha)$  such that  $UTU^* = M_t$  is the multiplication operator by the independent variable on  $L^2(\mathbb{P})$ ,  $U_\alpha(T + S_\alpha)U_\alpha^* = M_s$  is the multiplication operator by the independent variable on  $L^2(\mathbb{P}_\alpha)$ , and one has both  $(U\varphi)(t) = 1$  on  $\mathbb{R}$  and  $(U_\alpha\varphi)(s) = 1$  on  $\mathbb{R}$ . Clearly, the operators  $U$  and



$U_\alpha$  are uniquely determined by these properties. By [19, Theorem 2.1], the unitary operator  $V_\alpha := U_\alpha U^* : L^2(\mathbb{P}) \rightarrow L^2(\mathbb{P}_\alpha)$  is given by

$$(4.1) \quad (V_\alpha f)(x) = f(x) - \alpha \int \frac{f(x) - f(t)}{x - t} d\mathbb{P}(t)$$

for all continuously differentiable functions  $f : \mathbb{R} \rightarrow \mathbb{C}$  with compact support. For the rest of this subsection, we suppose that  $V_\alpha$  satisfies (4.1). Without loss of generality, we may further assume that  $T$  is already the multiplication operator by the independent variable on  $L^2(\mathbb{P})$ , i. e., we identify  $\mathfrak{H}$  with  $L^2(\mathbb{P})$ ,  $T$  with  $UTU^*$ , and  $T + S_\alpha$  with  $U(T + S_\alpha)U^*$ .

In order to prove Theorem 4.3, we need the following lemma.

**Lemma 4.4.** *Let  $\lambda \in \mathbb{R} \setminus \{\max \sigma_{\text{ess}}(T)\}$ . Then one has that the dimension of  $(\text{Ran } P_\lambda) \cap (\text{Ran } Q_\lambda)$  is*

- (1) *infinite if and only if  $\lambda > \max \sigma_{\text{ess}}(T)$ .*
- (2) *zero if and only if  $\lambda < \max \sigma_{\text{ess}}(T)$ .*

*Proof.* The idea of this proof is essentially due to the author's supervisor, Vadim Kostykin.

The well-known fact (see, e. g., [33, Example 5.4]) that  $\text{supp } \mathbb{P}_\alpha = \sigma(T + S_\alpha)$  implies that the cardinality of  $(\lambda, \infty) \cap \text{supp } \mathbb{P}_\alpha$  is infinite [resp. finite] if and only if  $\lambda < \max \sigma_{\text{ess}}(T)$  [resp.  $\lambda > \max \sigma_{\text{ess}}(T)$ ].

**Case 1.** The cardinality of  $(\lambda, \infty) \cap \text{supp } \mathbb{P}_\alpha$  is finite.

Since  $\lambda > \max \sigma_{\text{ess}}(T)$ , it follows that

$$\dim \text{Ran } E_{[\lambda, \infty)}(T + S_\alpha) < \infty \quad \text{and} \quad \dim \text{Ran } E_{[\lambda, \infty)}(T) < \infty.$$

Therefore,  $\text{Ran } E_{(-\infty, \lambda)}(T + S_\alpha) \cap \text{Ran } E_{(-\infty, \lambda)}(T)$  is infinite dimensional.

**Case 2.** The cardinality of  $(\lambda, \infty) \cap \text{supp } \mathbb{P}_\alpha$  is infinite.

If  $\lambda \leq \min \sigma(T)$  or  $\lambda \leq \min \sigma(T + S_\alpha)$ , then  $(\text{Ran } P_\lambda) \cap (\text{Ran } Q_\lambda) = \{0\}$ , as claimed. Now suppose that  $\lambda > \min \sigma(T)$  and  $\lambda > \min \sigma(T + S_\alpha)$ .

Let  $f \in (\text{Ran } P_\lambda) \cap (\text{Ran } Q_\lambda)$ . Then one has

$$f(x) = 0 \text{ for } \mathbb{P}\text{-almost all } x \geq \lambda \quad \text{and} \quad (V_\alpha f)(x) = 0 \text{ for } \mathbb{P}_\alpha\text{-almost all } x \geq \lambda.$$

Choose a representative  $\tilde{f}$  in the equivalence class of  $f$  such that  $\tilde{f}(x) = 0$  for all  $x \geq \lambda$ . Let  $r \in \left(0, \frac{\max \sigma_{\text{ess}}(T) - \lambda}{3}\right)$ . According to [14, Corollary 6.4 (a)] and the fact that  $\mathbb{P}$  is a finite Borel measure on  $\mathbb{R}$ , we know that the set of continuously differentiable scalar-valued functions on  $\mathbb{R}$  with compact support is dense in  $L^2(\mathbb{P})$  with respect to  $\|\cdot\|_{L^2(\cdot)}$ . Thus, a standard mollifier argument shows that we can choose continuously differentiable functions  $\tilde{f}_n : \mathbb{R} \rightarrow \mathbb{C}$  with compact support such that

$$\|\tilde{f}_n - \tilde{f}\|_{L^2(\cdot)} < 1/n \quad \text{and} \quad \tilde{f}_n(x) = 0 \text{ for all } x \geq \lambda + r, \quad n \in \mathbb{N}.$$

In particular, we may insert  $\tilde{f}_n$  into Formula (4.1) and obtain

$$(V_\alpha \tilde{f}_n)(x) = \alpha \int_{(-\infty, \lambda+r)} \frac{\tilde{f}_n(t)}{x-t} d\mathbb{P}(t) \quad \text{for all } x \geq \lambda + 2r.$$

It is readily seen that

$$(Bg)(x) := \int_{(-\infty, \lambda+r)} \frac{g(t)}{x-t} d\mathbb{P}(t), \quad x \geq \lambda + 2r,$$

defines a bounded operator  $B : L^2(\mathbb{1}_{(-\infty, \lambda+r)} d\mu) \rightarrow L^2(\mathbb{1}_{[\lambda+2r, \infty)} d\mu_\alpha)$  with operator norm  $\leq 1/r$ . It is now easy to show that

$$(*) \quad \int_{(-\infty, \lambda]} \frac{\tilde{f}(t)}{x-t} d\mu(t) = 0 \quad \text{for } \mu_\alpha\text{-almost all } x \geq \lambda + 2r.$$

As  $r \in \left(0, \frac{\max \sigma_{\text{ess}}(T) - \lambda}{3}\right)$  in  $(*)$  was arbitrary, we get that

$$\int_{(-\infty, \lambda]} \frac{\tilde{f}(t)}{x-t} d\mu(t) = 0 \quad \text{for } \mu_\alpha\text{-almost all } x > \lambda.$$

From now on, we may assume without loss of generality that  $\tilde{f}$  is real-valued.

Consider the holomorphic function from  $\mathbb{C} \setminus (-\infty, \lambda]$  to  $\mathbb{C}$  defined by

$$z \mapsto \int_{(-\infty, \lambda]} \frac{\tilde{f}(t)}{z-t} d\mu(t).$$

Since  $\lambda < \max \sigma_{\text{ess}}(T)$ , the identity theorem for holomorphic functions implies that

$$\int_{(-\infty, \lambda]} \frac{\tilde{f}(t)}{z-t} d\mu(t) = 0 \quad \text{for all } z \in \mathbb{C} \setminus (-\infty, \lambda].$$

This yields

$$(**) \quad \int_{(-\infty, \lambda]} \frac{\tilde{f}(t)}{(x-t)^2 + y^2} d\mu(t) = 0 \quad \text{for all } x \in \mathbb{R}, y > 0.$$

Consider the positive finite Borel measure  $\nu_1 : \mathcal{B}(\mathbb{R}) \rightarrow [0, \infty)$  and the finite signed Borel measure  $\nu_2 : \mathcal{B}(\mathbb{R}) \rightarrow \mathbb{R}$  defined by

$$\nu_1(\Omega) := \int_{\Omega \cap (-\infty, \lambda]} d\mu(t), \quad \nu_2(\Omega) := \int_{\Omega \cap (-\infty, \lambda]} \tilde{f}(t) d\mu(t);$$

note that  $\tilde{f}$  belongs to  $L^1(\mu)$ .

Denote by  $p_{\nu_j} : \{x + iy \in \mathbb{C} : x \in \mathbb{R}, y > 0\} \rightarrow \mathbb{R}$  the Poisson transform of  $\nu_j$ ,

$$p_{\nu_j}(x + iy) := y \int_{\mathbb{R}} \frac{d\nu_j(t)}{(x-t)^2 + y^2}, \quad x \in \mathbb{R}, y > 0, j = 1, 2.$$

It follows from  $(**)$  that

$$p_{\nu_2}(x + iy) = 0 \quad \text{for all } x \in \mathbb{R}, y > 0.$$

Furthermore, since  $\nu_1$  is not the trivial measure, one has

$$p_{\nu_1}(x + iy) > 0 \quad \text{for all } x \in \mathbb{R}, y > 0.$$

Now [12, Proposition 2.2] implies that

$$0 = \lim_{y \searrow 0} \frac{p_{\nu_2}(x + iy)}{p_{\nu_1}(x + iy)} = \tilde{f}(x) \quad \text{for } \mu\text{-almost all } x \leq \lambda.$$

Hence  $\tilde{f}(x) = 0$  for  $\mu$ -almost all  $x \in \mathbb{R}$ . We conclude that  $(\text{Ran } P_\lambda) \cap (\text{Ran } Q_\lambda)$  is trivial. This finishes the proof.  $\square$

Analogously, one shows that the following lemma holds true.

**Lemma 4.5.** *Let  $\lambda \in \mathbb{R} \setminus \{\min \sigma_{\text{ess}}(T)\}$ . Then one has that the dimension of  $(\text{Ker } P_\lambda) \cap (\text{Ker } Q_\lambda)$  is*

- (1) *infinite if and only if  $\lambda < \min \sigma_{\text{ess}}(T)$ .*
- (2) *zero if and only if  $\lambda > \min \sigma_{\text{ess}}(T)$ .*

**Remark.** The proof of Lemma 4.4 does not work if  $T$  is unbounded. To see this, consider the case where the essential spectrum of  $T$  is bounded from above and  $T$  has infinitely many isolated eigenvalues greater than  $\max \sigma_{\text{ess}}(T)$ .

*Proof of Theorem 4.3.* Taken together, Lemmas 4.4 and 4.5 imply Theorem 4.3.  $\square$

## 5. ON NON-INVERTIBILITY OF $D(\lambda)$

In this section, the self-adjoint operator  $T$  is assumed to be bounded. The main purpose of this section is to establish the following theorem.

**Theorem 5.1.** *Let  $S : \mathfrak{H} \rightarrow \mathfrak{H}$  be a compact self-adjoint operator. Then the following assertions hold true.*

- (1) *If  $\lambda \in \mathbb{R} \setminus \sigma_{\text{ess}}(T)$ , then  $D(\lambda)$  is a compact operator. In particular, zero belongs to the essential spectrum of  $D(\lambda)$ .*
- (2) *Zero belongs to the essential spectrum of  $D(\lambda)$  for all but at most countably many  $\lambda$  in  $\sigma_{\text{ess}}(T)$ .*

Note that we cannot exclude the case that the exceptional set is dense in  $\sigma_{\text{ess}}(T)$ .

**Remark.** Martínez-Avendaño and Treil have shown “that given any compact subset of the complex plane containing zero, there exists a Hankel operator having this set as its spectrum” (see [21, p. 83]). Thus, Theorem 5.1 and [21, Theorem 1.1] lead to the following result:

*for all  $\lambda$  in  $\mathbb{R}$  except for at most countably many  $\lambda$  in  $\sigma_{\text{ess}}(T)$ , there exists a Hankel operator  $\Gamma_\lambda$  such that  $\sigma(\Gamma_\lambda) = \sigma(D(\lambda))$ .*

First, we will prove Theorem 5.1 in the case when the range of  $S$  is finite dimensional. If  $S$  is compact and the range of  $S$  is infinite dimensional, then the proof has to be modified.

**5.1. The case when the range of  $S$  is finite dimensional.** Throughout this subsection, we consider a self-adjoint finite rank operator

$$S = \sum_{j=1}^N \alpha_j \langle \cdot, \varphi_j \rangle \varphi_j : \mathfrak{H} \rightarrow \mathfrak{H}, \quad N \in \mathbb{N},$$

where  $\varphi_1, \dots, \varphi_N$  form an orthonormal system in  $\mathfrak{H}$  and  $\alpha_1, \dots, \alpha_N$  are nonzero real numbers.

Note that if there exists  $\lambda_0$  in  $\mathbb{R}$  such that

$$\dim \text{Ran } E_{\{\lambda_0\}}(T) = \infty \quad \text{or} \quad \dim \text{Ran } E_{\{\lambda_0\}}(T + S) = \infty,$$

then  $\dim \text{Ker } D(\lambda) = \infty$  (see the proof of Proposition 4.1 (1) above) and hence  $0 \in \sigma_{\text{ess}}(D(\lambda))$  for all  $\lambda \in \mathbb{R}$ .

Define the sets  $\mathcal{M}(X)$ ,  $\mathcal{M}_-(X)$ , and  $\mathcal{M}_+(X)$  by

$$\begin{aligned} \mathcal{M}(X) &:= \{\lambda \in \sigma_{\text{ess}}(X) : \text{there exist } \lambda_k^\pm \text{ in } \sigma(X) \text{ such that } \lambda_k^- \nearrow \lambda, \lambda_k^+ \searrow \lambda\}, \\ \mathcal{M}_-(X) &:= \{\lambda \in \sigma_{\text{ess}}(X) : \text{there exist } \lambda_k^- \text{ in } \sigma(X) \text{ such that } \lambda_k^- \nearrow \lambda\} \setminus \mathcal{M}(X), \end{aligned}$$

$\mathcal{M}_+(X) := \{\lambda \in \sigma_{\text{ess}}(X) : \text{there exist } \lambda_k^+ \text{ in } \sigma(X) \text{ such that } \lambda_k^+ \searrow \lambda\} \setminus \mathcal{M}(X)$ , where  $X = T$  or  $X = T + S$ . The following well-known result shows that these sets do not depend on whether  $X = T$  or  $X = T + S$ .

**Lemma 5.2** (see [1, Proposition 2.1]; see also [3, p. 83]). *Let  $A$  and  $B$  be bounded self-adjoint operators acting on  $\mathfrak{H}$ . If  $N := \dim \text{Ran } B$  is in  $\mathbb{N}$  and  $\mathcal{I} \subset \mathbb{R}$  is a nonempty interval contained in the resolvent set of  $A$ , then  $\mathcal{I}$  contains no more than  $N$  eigenvalues of the operator  $A + B$  (taking into account their multiplicities).*

In view of this lemma and the fact that the essential spectrum is invariant under compact perturbations, we will write  $\mathcal{M}$  instead of  $\mathcal{M}(X)$ ,  $\mathcal{M}_+$  instead of  $\mathcal{M}_+(X)$ , and  $\mathcal{M}_-$  instead of  $\mathcal{M}_-(X)$ , where  $X = T$  or  $X = T + S$ .

**Lemma 5.3.** *Let  $\lambda \in \mathbb{R} \setminus (\mathcal{M} \cup \mathcal{M}_-)$ . Then  $D(\lambda)$  is a trace class operator.*

*Proof.* There exists an infinitely differentiable function  $\psi : \mathbb{R} \rightarrow \mathbb{R}$  with compact support such that

$$E_{(-\infty, \lambda)}(T + S) - E_{(-\infty, \lambda)}(T) = \psi(T + S) - \psi(T).$$

Combine [4, p. 156, Equation (8.3)] with [25, p. 532] and [25, Theorem 2], and it follows that  $D(\lambda)$  is a trace class operator.  $\square$

An analogous proof shows that  $D(\lambda)$  is a trace class operator for  $\lambda$  in  $\mathcal{M}_-$ , provided that  $E_{\{\lambda\}}(T + S) - E_{\{\lambda\}}(T)$  is of trace class.

**Proposition 5.4.** *One has  $0 \in \sigma_{\text{ess}}(D(\lambda))$  for all but at most countably many  $\lambda \in \mathbb{R}$ .*

In the proof of Proposition 5.4, we will use the notion of weak convergence for sequences of probability measures.

**Definition 5.5.** Let  $\mathcal{E}$  be a metric space. A sequence  $\nu_1, \nu_2, \dots$  of Borel probability measures on  $\mathcal{E}$  is said to converge *weakly* to a Borel probability measure  $\nu$  on  $\mathcal{E}$  if

$$\lim_{n \rightarrow \infty} \int f d\nu_n = \int f d\nu \quad \text{for every bounded continuous function } f : \mathcal{E} \rightarrow \mathbb{R}.$$

If  $\nu_1, \nu_2, \dots$  converges weakly to  $\nu$ , then we shall write  $\nu_n \xrightarrow{w} \nu, n \rightarrow \infty$ .

*Proof of Proposition 5.4.* First, we note that if  $\lambda < \min(\sigma(T) \cup \sigma(T + S))$  or  $\lambda > \max(\sigma(T) \cup \sigma(T + S))$ , then  $D(\lambda)$  is the zero operator, and there is nothing to show. So let us henceforth assume that  $\lambda \geq \min(\sigma(T) \cup \sigma(T + S))$  and  $\lambda \leq \max(\sigma(T) \cup \sigma(T + S))$ .

The idea of the proof is to apply Weyl's criterion (see, e. g., [33, Proposition 8.11]) to a suitable sequence of normed vectors. In this proof, we denote by  $\|g\|_{\infty, \mathcal{K}}$  the supremum norm of a function  $g : \mathcal{K} \rightarrow \mathbb{R}$ , where  $\mathcal{K}$  is a compact subset of  $\mathbb{R}$ , and by  $\|A\|_{\text{op}}$  the usual operator norm of an operator  $A : \mathfrak{H} \rightarrow \mathfrak{H}$ .

Choose a sequence  $(x_n)_{n \in \mathbb{N}}$  of normed vectors in  $\mathfrak{H}$  such that

$$\begin{aligned} x_1 &\perp \{\varphi_k : k = 1, \dots, N\}, \quad x_2 \perp \{x_1, \varphi_k, T\varphi_k : k = 1, \dots, N\}, \quad \dots, \\ x_n &\perp \{x_1, \dots, x_{n-1}, T^j \varphi_k : j \in \mathbb{N}_0, j \leq n-1, k = 1, \dots, N\}, \quad \dots \end{aligned}$$

Consider sequences of Borel probability measures  $(\nu_n)_{n \in \mathbb{N}}$  and  $(\tilde{\nu}_n)_{n \in \mathbb{N}}$  that are defined as follows:

$$\nu_n(\Omega) := \langle E_\Omega(T)x_n, x_n \rangle, \quad \tilde{\nu}_n(\Omega) := \langle E_\Omega(T + S)x_n, x_n \rangle, \quad \Omega \in \mathcal{B}(\mathbb{R}).$$

It is easy to see that by Prohorov's theorem (see, e. g., [23, Proposition 7.2.3]), there exist a subsequence of a subsequence of  $(x_n)_{n \in \mathbb{N}}$  and Borel probability measures  $\nu$  and  $\tilde{\nu}$  with support contained in  $\sigma(T)$  and  $\sigma(T + S)$ , respectively, such that

$$\nu_{n_k} \xrightarrow{w} \nu \text{ as } k \rightarrow \infty \quad \text{and} \quad \tilde{\nu}_{n_{k_\ell}} \xrightarrow{w} \tilde{\nu} \text{ as } \ell \rightarrow \infty.$$

Due to this observation, we consider the sequences  $(x_{n_{k_\ell}})_{\ell \in \mathbb{N}}$ ,  $(\nu_{n_{k_\ell}})_{\ell \in \mathbb{N}}$ , and  $(\tilde{\nu}_{n_{k_\ell}})_{\ell \in \mathbb{N}}$  which will be denoted again by  $(x_n)_{n \in \mathbb{N}}$ ,  $(\nu_n)_{n \in \mathbb{N}}$ , and  $(\tilde{\nu}_n)_{n \in \mathbb{N}}$ .

Put  $\mathcal{N}_T := \{\mu \in \mathbb{R} : \nu(\{\mu\}) > 0\}$  and  $\mathcal{N}_{T+S} := \{\mu \in \mathbb{R} : \tilde{\nu}(\{\mu\}) > 0\}$ . Then the set  $\mathcal{N}_T \cup \mathcal{N}_{T+S}$  is at most countable. Consider the case where  $\lambda$  does not belong to  $\mathcal{N}_T \cup \mathcal{N}_{T+S}$ . Define  $\xi := \min \{\min \sigma(T), \min \sigma(T + S)\} - 1$ . Consider the continuous functions  $f_m : \mathbb{R} \rightarrow \mathbb{R}$ ,  $m \in \mathbb{N}$ , that are defined by

$$f_m(t) := (1 + m(t - \xi)) \cdot \mathbb{1}_{[\xi - 1/m, \xi]}(t) + \mathbb{1}_{(\xi, \lambda)}(t) + (1 - m(t - \lambda)) \cdot \mathbb{1}_{[\lambda, \lambda + 1/m]}(t).$$

The figure below shows (qualitatively) the graph of  $f_m$ .



FIGURE 1. The graph of  $f_m$ .

For all  $m \in \mathbb{N}$ , choose polynomials  $p_{m,k}$  such that

$$(5.1) \quad \|f_m - p_{m,k}\|_{\infty, \mathcal{K}} \rightarrow 0 \quad \text{as } k \rightarrow \infty,$$

where  $\mathcal{K} := [\min(\sigma(T) \cup \sigma(T + S)) - 10, \max(\sigma(T) \cup \sigma(T + S)) + 10]$ .

By construction of  $(x_n)_{n \in \mathbb{N}}$ , one has

$$(5.2) \quad p_{m,k}(T + S)x_n = p_{m,k}(T)x_n \quad \text{for all } n > \text{degree of } p_{m,k}.$$

For all  $m \in \mathbb{N}$ , the function  $|\mathbb{1}_{(-\infty, \lambda)} - f_m|^2$  is bounded, measurable, and continuous except for a set of both  $\nu$ -measure zero and  $\tilde{\nu}$ -measure zero.

Now (5.2) and the Portmanteau theorem (see, e. g., [13, Theorem 13.16 (i) and (iii)]) imply

$$\begin{aligned} & \limsup_{n \rightarrow \infty} \|(E_{(-\infty, \lambda)}(T + S) - E_{(-\infty, \lambda)}(T))x_n\| \\ & \leq \left( \int_{\mathbb{R}} |\mathbb{1}_{(-\infty, \lambda)}(t) - f_m(t)|^2 d\nu(t) \right)^{1/2} \\ & \quad + \left( \int_{\mathbb{R}} |\mathbb{1}_{(-\infty, \lambda)}(s) - f_m(s)|^2 d\tilde{\nu}(s) \right)^{1/2} \\ & \quad + \|f_m(T) - p_{m,k}(T)\|_{\text{op}} \\ & \quad + \|f_m(T + S) - p_{m,k}(T + S)\|_{\text{op}} \end{aligned}$$

for all  $m \in \mathbb{N}$  and all  $k \in \mathbb{N}$ . First, we send  $k \rightarrow \infty$  and then we take the limit  $m \rightarrow \infty$ . As  $m \rightarrow \infty$ , the sequence  $|\mathbb{1}_{(-\infty, \lambda)} - f_m|^2$  converges to zero pointwise almost everywhere with respect to both  $\nu$  and  $\tilde{\nu}$ . Now (5.1) and the dominated convergence theorem imply

$$\lim_{n \rightarrow \infty} \|(E_{(-\infty, \lambda)}(T + S) - E_{(-\infty, \lambda)}(T))x_n\| = 0.$$

Recall that  $(x_n)_{n \in \mathbb{N}}$  is an orthonormal sequence. Thus, an application of Weyl's criterion (see, e. g., [33, Proposition 8.11]) concludes the proof.  $\square$

**Remark.** If  $T$  is unbounded, then the spectrum of  $T$  is unbounded, so that the proof of Proposition 5.4 does not work. For instance, we used the compactness of the spectrum in order to uniformly approximate  $f_m$  by polynomials.

Moreover, it is unclear whether an orthonormal sequence  $(x_n)_{n \in \mathbb{N}}$  as in the proof of Proposition 5.4 can be found in the domain of  $T$ .

**5.2. The case when the range of  $S$  is infinite dimensional.** In this subsection, we suppose that  $S$  is compact and the range of  $S$  is infinite dimensional. The following lemma is easily shown.

**Lemma 5.6.** *Let  $\lambda \in \mathbb{R} \setminus \sigma_{\text{ess}}(T)$ . Then  $D(\lambda)$  is compact.*

Furthermore, Proposition 5.4 still holds when  $S$  is compact and the range of  $S$  is infinite dimensional. To see this, we need to modify two steps of the proof of Proposition 5.4.

Let us write  $S = \sum_{j=1}^{\infty} \alpha_j \langle \cdot, \varphi_j \rangle \varphi_j$ , where  $\varphi_1, \varphi_2, \dots$  is an orthonormal system in  $\mathfrak{H}$  and  $\alpha_1, \alpha_2, \dots$  are nonzero real numbers.

(1) In contrast to the proof of Proposition 5.4 above, we choose an orthonormal sequence  $x_1, x_2, \dots$  in  $\mathfrak{H}$  as follows:

$$\begin{aligned} x_1 &\perp \varphi_1, \quad x_2 \perp \{x_1, \varphi_1, \varphi_2, T\varphi_1, T\varphi_2\}, \quad \dots, \\ x_n &\perp \{x_1, \dots, x_{n-1}, \varphi_k, T\varphi_k, \dots, T^{n-1}\varphi_k : k = 1, \dots, n\}, \quad \dots \end{aligned}$$

By construction, one has

$$p(T + F_\ell)x_n = p(T)x_n \quad \text{for all } n > \max(\ell, \text{degree of } p),$$

where  $p$  is a polynomial,  $\ell \in \mathbb{N}$ , and  $F_\ell := \sum_{j=1}^{\ell} \alpha_j \langle \cdot, \varphi_j \rangle \varphi_j$ .

(2) We continue as in the proof of Proposition 5.4 above and estimate as follows:

$$\begin{aligned} &\limsup_{n \rightarrow \infty} \|(E_{(-\infty, \lambda)}(T + S) - E_{(-\infty, \lambda)}(T))x_n\| \\ &\leq \left( \int_{\mathbb{R}} |\mathbb{1}_{(-\infty, \lambda)}(t) - f_m(t)|^2 d\nu(t) \right)^{1/2} \\ &\quad + \left( \int_{\mathbb{R}} |\mathbb{1}_{(-\infty, \lambda)}(s) - f_m(s)|^2 d\tilde{\nu}(s) \right)^{1/2} \\ &\quad + \|f_m(T) - p_{m,k}(T)\|_{\text{op}} \\ &\quad + \|f_m(T + S) - p_{m,k}(T + S)\|_{\text{op}} \\ &\quad + \|p_{m,k}(T + S) - p_{m,k}(T + F_\ell)\|_{\text{op}} \end{aligned}$$

for all  $k, \ell, m \in \mathbb{N}$ , where  $\|\cdot\|_{\text{op}}$  denotes the operator norm. It is well known that the operators  $F_\ell$  uniformly approximate the operator  $S$  as  $\ell$  tends to infinity. Therefore,  $\|p_{m,k}(T + S) - p_{m,k}(T + F_\ell)\|_{\text{op}} \rightarrow 0$  as  $\ell \rightarrow \infty$ .

Analogously to the proof of Proposition 5.4 above, it follows that

$$\lim_{n \rightarrow \infty} \|(E_{(-\infty, \lambda)}(T + S) - E_{(-\infty, \lambda)}(T))x_n\| = 0.$$

Hence, we have shown that zero belongs to the essential spectrum of  $D(\lambda)$  for all but at most countably many  $\lambda \in \mathbb{R}$ .

**5.3. Proof of Theorem 5.1.** Taken together, Lemma 5.3 and Proposition 5.4 show that Theorem 5.1 holds whenever the range of  $S$  is finite dimensional.

In the preceding subsection, we have shown that Theorem 5.1 also holds when  $S$  is compact and the range of  $S$  is infinite dimensional.

Now the proof is complete.  $\square$

**5.4. The smooth situation.** In order to apply a result of Pushnitski [27] to  $D(\lambda)$ , we check the corresponding assumptions stated in [27, p. 228].

First, define the compact self-adjoint operator  $G := |S|^{\frac{1}{2}} : \mathfrak{H} \rightarrow \mathfrak{H}$  and the bounded self-adjoint operator  $S_0 := \text{sign}(S) : \mathfrak{H} \rightarrow \mathfrak{H}$ . Obviously, one has  $S = G^* S_0 G$ . Define the operator-valued functions  $h_0$  and  $h$  on  $\mathbb{R}$  by

$$h_0(\lambda) = GE_{(-\infty, \lambda)}(T)G^*, \quad h(\lambda) = GE_{(-\infty, \lambda)}(T + S)G^*, \quad \lambda \in \mathbb{R}.$$

In order to fulfill [27, Hypothesis 1.1], we need the following assumptions.

**Hypothesis.** Suppose that there exists an open interval  $\delta$  contained in the absolutely continuous spectrum of  $T$ . Next, we assume that the derivatives

$$\dot{h}_0(\lambda) = \frac{d}{d\lambda} h_0(\lambda) \quad \text{and} \quad \dot{h}(\lambda) = \frac{d}{d\lambda} h(\lambda)$$

exist in operator norm for all  $\lambda \in \delta$ , and that the maps  $\delta \ni \lambda \mapsto \dot{h}_0(\lambda)$  and  $\delta \ni \lambda \mapsto \dot{h}(\lambda)$  are Hölder continuous (with some positive exponent) in the operator norm.

Now [27, Theorem 1.1] yields that for all  $\lambda \in \delta$ , there exists a nonnegative real number  $a$  such that

$$\sigma_{\text{ess}}(D(\lambda)) = [-a, a].$$

The number  $a$  depends on  $\lambda$  and can be expressed in terms of the scattering matrix for the pair  $T, T + S$ , see [27, Formula (1.3)].

**Example 5.7.** Again, consider Kreĭn's example [16, pp. 622–624]. That is,  $\mathfrak{H} = L^2(0, \infty)$ , the initial operator  $T = A_0$  is the integral operator from Example 4.2, and  $S = \langle \cdot, \varphi \rangle \varphi$  with  $\varphi(x) = e^{-x}$ . Put  $\delta = (0, 1)$ . Then Pushnitski has shown in [27, Subsection 1.3] that, by [27, Theorem 1.1], one has  $\sigma_{\text{ess}}(D(\lambda)) = [-1, 1]$  for all  $0 < \lambda < 1$ .

In particular, the operator  $D(\lambda)$  fulfills condition (C2) in Theorem 2.2 for all  $0 < \lambda < 1$ .

## 6. PROOF OF THE MAIN RESULTS

This section is devoted to the proof of Theorem 2 and Theorem 3. First, we need to show two auxiliary results.

### 6.1. Two auxiliary results.

**Proposition 6.1.** Let  $T$  and  $S = \langle \cdot, \varphi \rangle \varphi$  be a bounded self-adjoint operator and a self-adjoint operator of rank one acting on  $\mathfrak{H}$ , respectively.

- (1) The operator  $D(\lambda)$  is unitarily equivalent to a self-adjoint Hankel operator of finite rank for all  $\lambda$  in  $\mathbb{R} \setminus [\min \sigma_{\text{ess}}(T), \max \sigma_{\text{ess}}(T)]$ .
- (2) Suppose that  $\varphi$  is cyclic for  $T$ . Then  $D(\lambda)$  is unitarily equivalent to a bounded self-adjoint Hankel operator for all  $\lambda$  in  $\mathbb{R} \setminus \sigma_{\text{ess}}(T)$  and for all but at most countably many  $\lambda$  in  $\sigma_{\text{ess}}(T)$ .



*Proof.* (1) follows easily from Lemma 3.1 and Theorem 2.2, because

$$\begin{aligned} D(\lambda) &= E_{(-\infty, \lambda)}(T + S) - E_{(-\infty, \lambda)}(T) \\ &= E_{[\lambda, \infty)}(T) - E_{[\lambda, \infty)}(T + S) \end{aligned}$$

is a finite rank operator for all  $\lambda$  in  $(-\infty, \min \sigma_{\text{ess}}(T)) \cup (\max \sigma_{\text{ess}}(T), \infty)$ .

(2) is a direct consequence of Lemma 3.1, Theorem 4.3, Theorem 5.1, and Theorem 2.2.

This concludes the proof.  $\square$

**Lemma 6.2.** *The statements of Theorem 3 hold if  $T$  is additionally assumed to be bounded.*

*Proof.* Let  $\lambda \in \mathbb{R}$ . It follows from Halmos' decomposition (see [10]) of  $\mathfrak{H}$  with respect to the orthogonal projections  $E_{(-\infty, \lambda)}(T + S)$  and  $E_{(-\infty, \lambda)}(T)$  that we obtain the following orthogonal decomposition of  $\mathfrak{H}$  with respect to  $D(\lambda)$ :

$$\mathfrak{H} = \left( \text{Ker } D(\lambda) \right) \oplus \left( \text{Ran } E_{\{1\}}(D(\lambda)) \right) \oplus \left( \text{Ran } E_{\{-1\}}(D(\lambda)) \right) \oplus \mathfrak{H}_g^{(\lambda)}.$$

Here  $\mathfrak{H}_g^{(\lambda)}$  is the orthogonal complement of

$$\tilde{\mathfrak{H}}^{(\lambda)} := \left( \text{Ker } D(\lambda) \right) \oplus \left( \text{Ran } E_{\{1\}}(D(\lambda)) \right) \oplus \left( \text{Ran } E_{\{-1\}}(D(\lambda)) \right)$$

in  $\mathfrak{H}$ . Clearly,  $\mathfrak{H}_g^{(\lambda)}$  is reducing for the operator  $D(\lambda)$ . It follows from Lemma 2.1 that  $D(\lambda)|_{\mathfrak{H}_g^{(\lambda)}}$  is unitarily equivalent to  $-D(\lambda)|_{\tilde{\mathfrak{H}}^{(\lambda)}}$ .

It is elementary to show that there exists a compact self-adjoint block diagonal operator  $\tilde{K}_\lambda \oplus 0$  on  $\tilde{\mathfrak{H}}^{(\lambda)} \oplus \mathfrak{H}_g^{(\lambda)}$  with the following properties:

- $\tilde{K}_\lambda \oplus 0$  fulfills assertion (2) in Theorem 3.
- the range of  $\tilde{K}_\lambda \oplus 0$  is infinite dimensional if and only if one of the closed subspaces  $\text{Ran } E_{\{1\}}(D(\lambda))$ ,  $\text{Ran } E_{\{-1\}}(D(\lambda))$  is finite dimensional and the other one is infinite dimensional.
- the kernel of  $D(\lambda) - (\tilde{K}_\lambda \oplus 0)$  is either trivial or infinite dimensional.
- if  $\tilde{\mathfrak{H}}^{(\lambda)} \neq \{0\}$ , then the spectrum of  $D(\lambda)|_{\tilde{\mathfrak{H}}^{(\lambda)}} - \tilde{K}_\lambda$  is contained in the interval  $[-1, 1]$  and consists only of eigenvalues. Moreover, the dimensions of  $\text{Ran } E_{\{t\}}(D(\lambda)|_{\tilde{\mathfrak{H}}^{(\lambda)}} - \tilde{K}_\lambda)$  and  $\text{Ran } E_{\{-t\}}(D(\lambda)|_{\tilde{\mathfrak{H}}^{(\lambda)}} - \tilde{K}_\lambda)$  differ by at most one, for all  $0 < t \leq 1$ .

The block diagonal operator  $\tilde{K}_\lambda \oplus 0$  serves as a correction term for  $D(\lambda)$ . In particular, no correction term is needed if  $\tilde{\mathfrak{H}}^{(\lambda)} = \{0\}$ .

Theorem 5.1 and the invariance of the essential spectrum under compact perturbations imply that zero belongs to the essential spectrum of  $D(\lambda) - (\tilde{K}_\lambda \oplus 0)$  for all  $\lambda$  in  $\mathbb{R}$  except for at most countably many  $\lambda$  in  $\sigma_{\text{ess}}(T)$ .

Therefore, an application of [22, Theorem 1] yields that  $D(\lambda) - (\tilde{K}_\lambda \oplus 0)$  is unitarily equivalent to a bounded self-adjoint Hankel operator  $\Gamma_\lambda$  on  $\ell^2(\mathbb{N}_0)$  for all  $\lambda$  in  $\mathbb{R}$  except for at most countably many  $\lambda$  in  $\sigma_{\text{ess}}(T)$ .

Thus, by the properties of  $\tilde{K}_\lambda \oplus 0$  listed above, the claim follows.  $\square$

**Remark.** If we consider  $E_{(-\infty, \lambda]}(T) - E_{(-\infty, \lambda]}(T + S)$ , the difference of the spectral projections associated with the closed interval  $(-\infty, \lambda]$  instead of the open interval  $(-\infty, \lambda)$ , then all assertions in Lemma 3.1, Proposition 4.1, Theorem 4.3, Theorem 5.1, Proposition 6.1, and Lemma 6.2 remain true. All proofs can easily be modified.

**6.2. The case when  $T$  is semibounded.** In this subsection, which is based on Kreĭn's approach in [16, pp. 622–623], we deal with the case when the self-adjoint operator  $T$  is semibounded *but not bounded*. As before, we write

$$D(\lambda) = E_{(-\infty, \lambda)}(T + S) - E_{(-\infty, \lambda)}(T)$$

if  $S$  is a compact self-adjoint operator and  $\lambda \in \mathbb{R}$ .

First, consider the case when  $T$  is bounded from below. Choose  $c \in \mathbb{R}$  such that

$$(6.1) \quad T + cI \geq 0 \quad \text{and} \quad T + S + cI \geq 0.$$

It suffices to consider  $D(\lambda)$  for  $\lambda \geq -c$ . Compute

$$\begin{aligned} D(\lambda) &= E_{[\lambda, \infty)}(T) - E_{[\lambda, \infty)}(T + S) \\ &= E_{(-\infty, \mu]}((T + (1 + c)I)^{-1}) - E_{(-\infty, \mu]}((T + S + (1 + c)I)^{-1}), \end{aligned}$$

where  $\mu = \frac{1}{\lambda + 1 + c}$ . By the second resolvent equation, one has

$$(T + S + (1 + c)I)^{-1} = (T + (1 + c)I)^{-1} - (T + S + (1 + c)I)^{-1}S(T + (1 + c)I)^{-1}.$$

The operator

$$(6.2) \quad S' := -(T + S + (1 + c)I)^{-1}S(T + (1 + c)I)^{-1}$$

is compact and self-adjoint. One can easily show that  $\text{rank } S' = \text{rank } S$ .

In particular, if  $S = \langle \cdot, \varphi \rangle \varphi$  has rank one and  $\varphi' := \frac{(T + (1 + c)I)^{-1}\varphi}{\|(T + (1 + c)I)^{-1}\varphi\|}$ , then there exists a number  $\alpha' \in \mathbb{R}$  such that  $S' = \alpha' \langle \cdot, \varphi' \rangle \varphi'$ .

We have shown:

**Lemma 6.3.** *Let  $T$  be a self-adjoint operator which is bounded from below but not bounded, let  $S$  be a compact self-adjoint operator, and let  $c$  be such that (6.1) holds. Then  $D(\lambda) = 0$  for all  $\lambda < -c$  and*

$$D(\lambda) = E_{(-\infty, \mu]}(T') - E_{(-\infty, \mu]}(T' + S') \quad \text{for all } \lambda \geq -c.$$

Here  $\mu = \frac{1}{\lambda + 1 + c}$ ,  $T' = (T + (1 + c)I)^{-1}$ , and  $S'$  is defined as in (6.2).

The case when  $T$  is bounded from below can now be pulled back to the bounded case, see the remark in Subsection 6.1 above.

**Proposition 6.4.** *Suppose that  $S = \langle \cdot, \varphi \rangle \varphi$  is a self-adjoint operator of rank one and that  $T$  is a self-adjoint operator which is bounded from below but not bounded. Assume further that the spectrum of  $T$  is not purely discrete and that the vector  $\varphi$  is cyclic for  $T$ . Then the kernel of  $D(\lambda)$  is trivial for all  $\lambda > \min \sigma_{\text{ess}}(T)$ .*

*Furthermore,  $D(\lambda)$  is unitarily equivalent to a bounded self-adjoint Hankel operator for all  $\lambda$  in  $\mathbb{R} \setminus \sigma_{\text{ess}}(T)$  and for all but at most countably many  $\lambda$  in  $\sigma_{\text{ess}}(T)$ .*

*Proof.* Let  $c$  be such that (6.1) holds.

It is easy to show that  $(T + (1 + c)I)^{-1}\varphi$  is cyclic for  $(T + (1 + c)I)^{-1}$  if  $\varphi$  is cyclic for  $T$ .

Furthermore, it is easy to show that the function  $x \mapsto \frac{1}{x + 1 + c}$  is one-to-one from  $\sigma_{\text{ess}}(T)$  onto  $\sigma_{\text{ess}}((T + (1 + c)I)^{-1}) \setminus \{0\}$ .

One has that  $\min \sigma_{\text{ess}}((T + (1 + c)I)^{-1}) = 0$  and, since the spectrum of  $T$  is not purely discrete,  $\max \sigma_{\text{ess}}((T + (1 + c)I)^{-1}) = \frac{1}{\lambda_0 + 1 + c}$ , where  $\lambda_0 := \min \sigma_{\text{ess}}(T)$ .

Therefore,  $\mu = \frac{1}{\lambda + 1 + c}$  belongs to the open interval  $(0, \frac{1}{\lambda_0 + 1 + c})$  if and only if  $\lambda > \lambda_0$ .

In view of Lemma 6.3 and the remark in Subsection 6.1 above, the claims follow.  $\square$

Moreover, standard computations show:

**Corollary 6.5.** *Suppose that  $S$  is a self-adjoint operator of finite rank  $N \in \mathbb{N}$  and that  $T$  is a self-adjoint operator which is bounded from below but not bounded. We obtain the same list of sufficient conditions for  $D(\lambda)$  to be unitarily equivalent to a bounded self-adjoint block-Hankel operator of order  $N$  with infinite dimensional kernel for all  $\lambda \in \mathbb{R}$  as in Proposition 4.1 above.*

*Proof.* Let  $X = T$  or  $X = T + S$  and let  $c$  be such that (6.1) holds. One has:

- The real number  $\lambda$  is an eigenvalue of  $X$  with multiplicity  $k \in \mathbb{N} \cup \{\infty\}$  if and only if  $\frac{1}{\lambda+1+c}$  is an eigenvalue of  $(X + (1+c)I)^{-1}$  with the same multiplicity  $k$ .
- The spectrum of the restricted operator  $X|_{\mathfrak{E}^\perp}$  has multiplicity at least  $N+1$  if and only if the spectrum of the restricted operator  $(X + (1+c)I)^{-1}|_{\mathfrak{E}^\perp}$  has multiplicity at least  $N+1$ , where  $\mathfrak{E} := \{x \in \mathfrak{H} : x \text{ is an eigenvector of } X\}$ .

In view of Lemma 6.3 and the remark in Subsection 6.1 above, the claim follows.  $\square$

In Proposition 6.4, we assumed that the spectrum of  $T$  is not purely discrete. Now consider the case when  $T$  has a purely discrete spectrum. By the invariance of the essential spectrum under compact perturbations, it is clear that the operator  $T + S$  also has a purely discrete spectrum, for all compact self-adjoint operators  $S$ . Moreover, since  $T$  is bounded from below, we know that  $T + S$  is bounded from below as well. Therefore, the range of  $D(\lambda)$  is finite dimensional for all  $\lambda \in \mathbb{R}$ , and in particular conditions (C1) and (C2) in Theorem 2.3 are fulfilled.

Combining this with Lemma 3.1 and the remark in Subsection 6.1 above, we have shown:

**Proposition 6.6.** *Suppose that  $S$  is a self-adjoint operator of finite rank  $N \in \mathbb{N}$  and that  $T$  is a bounded from below self-adjoint operator with a purely discrete spectrum. Then  $D(\lambda)$  is unitarily equivalent to a finite rank self-adjoint block-Hankel operator of order  $N$  for all  $\lambda \in \mathbb{R}$ .*

This proposition supports the idea that there is a structural correlation between the operator  $D(\lambda)$  and block-Hankel operators.

Now, consider the case when  $T$  is bounded from above. Choose  $c \in \mathbb{R}$  such that

$$T - cI \leq 0 \quad \text{and} \quad T + S - cI \leq 0.$$

It suffices to consider  $D(\lambda)$  for  $\lambda \leq c$ . Compute

$$\begin{aligned} D(\lambda) &= E_{(\mu, \infty)}((T + S - (1+c)I)^{-1}) - E_{(\mu, \infty)}((T - (1+c)I)^{-1}) \\ &= E_{(-\infty, \mu]}((T - (1+c)I)^{-1}) - E_{(-\infty, \mu]}((T + S - (1+c)I)^{-1}), \end{aligned}$$

where  $\mu = \frac{1}{\lambda - (1+c)}$ . By the second resolvent equation, one has

$$(T + S - (1+c)I)^{-1} = (T - (1+c)I)^{-1} - (T + S - (1+c)I)^{-1}S(T - (1+c)I)^{-1}.$$

The operator  $S'' := -(T + S - (1+c)I)^{-1}S(T - (1+c)I)^{-1}$  is compact and self-adjoint with  $\text{rank } S'' = \text{rank } S$ .

In particular, if  $S = \langle \cdot, \varphi \rangle \varphi$  has rank one and  $\varphi'' := \frac{(T - (1+c)I)^{-1} \varphi}{\|(T - (1+c)I)^{-1} \varphi\|}$ , then there exists a number  $\alpha'' \in \mathbb{R}$  such that  $S'' = \alpha'' \langle \cdot, \varphi'' \rangle \varphi''$ .

Now proceed analogously to the case when  $T$  is bounded from below.

It follows that Proposition 6.4 holds in the case when  $T$  is bounded from above but not bounded if we replace  $\lambda > \min \sigma_{\text{ess}}(T)$  by  $\lambda < \max \sigma_{\text{ess}}(T)$ .

Furthermore, Corollary 6.5 still holds if  $T$  is bounded from above but not bounded.

Obviously, Proposition 6.6 holds in the case when  $T$  is bounded from above.

**6.3. Proof of Theorem 2 and Theorem 3.** Let us first complete the proof of Theorem 2.

*Proof of Theorem 2.* If the operator  $T$  is bounded, then the statement of Theorem 2 follows from Proposition 6.1 and the discussion of Case 1 – Case 3 in Subsection 4.2 above.

Now suppose that  $T$  is bounded from below but not bounded and let  $c$  be such that (6.1) holds. First, assume that the spectrum of  $T$  is not purely discrete. If  $\varphi$  is cyclic for  $T$ , then the claim follows from Proposition 6.4. In the case when  $\varphi$  is not cyclic for  $T$ , we consider the bounded operator  $T'$  and the rank one operator  $S'$  defined as in Lemma 6.3 above. As we have noted in the proof of Proposition 6.4, it is easy to show that the function  $x \mapsto \frac{1}{x+1+c}$  is one-to-one from  $\sigma_{\text{ess}}(T)$  onto  $\sigma_{\text{ess}}(T') \setminus \{0\}$ . Now the statement of Theorem 2 follows from the remark in Subsection 6.1, Proposition 6.1, and the discussion of Case 1 – Case 3 in Subsection 4.2 above.

If  $T$  has a purely discrete spectrum, then Proposition 6.6 shows that  $D(\lambda)$  is unitarily equivalent to a finite rank self-adjoint Hankel operator for all  $\lambda \in \mathbb{R}$ .

If  $T$  is bounded from above but not bounded, then the proof runs analogously.

This finishes the proof.  $\square$

Now let us prove Theorem 3.

*Proof of Theorem 3.* In view of Lemma 6.2, it suffices to consider the case when  $T$  is semibounded but not bounded.

First, let  $T$  be bounded from below but not bounded and let  $c$  be such that (6.1) holds. Again, recall that the function  $x \mapsto \frac{1}{x+1+c}$  is one-to-one from  $\sigma_{\text{ess}}(T)$  onto  $\sigma_{\text{ess}}((T + (1+c)I)^{-1}) \setminus \{0\}$ . Now the statements of Theorem 3 follow from Lemma 6.3, the remark in Subsection 6.1 above, and Lemma 6.2.

If  $T$  is bounded from above but not bounded, then the proof runs analogously.  $\square$

## 7. SOME EXAMPLES

In this section, we apply the above theory in the context of operators that are of particular interest in various fields of (applied) mathematics, such as Schrödinger operators.

In any of the following examples, the operator  $D(\lambda)$  is unitarily equivalent to a bounded self-adjoint (block-) Hankel operator for all  $\lambda$  in  $\mathbb{R}$ .

First, we consider the case when  $T$  has a purely discrete spectrum.

**Example 7.1.** Let  $\mathfrak{H} = L^2(\mathbb{R}^n)$  and suppose that  $V \geq 0$  is in  $L^1_{\text{loc}}(\mathbb{R}^n)$  such that Lebesgue measure of  $\{x \in \mathbb{R}^n : 0 \leq V(x) < M\}$  is finite for all  $M > 0$ . Then the self-adjoint Schrödinger operator  $T \geq 0$  defined by the form sum of  $-\Delta$  and  $V$  has a purely discrete spectrum, see [37, Example 4.1]; see also [35, Theorem 1]. Therefore, if  $S$  is any self-adjoint operator of finite rank  $N$ , then Proposition 6.6 implies that  $D(\lambda)$  is unitarily equivalent to a finite rank self-adjoint block-Hankel operator of order  $N$  for all  $\lambda \in \mathbb{R}$ .

Next, consider the case when  $S = \langle \cdot, \varphi \rangle \varphi$  is of rank one and  $\varphi$  is cyclic for  $T$ .

**Example 7.2.** Once again, consider Kreĭn's example [16, pp. 622–624].

The operators  $T = A_0$  and  $T + \langle \cdot, \varphi \rangle \varphi = A_1$ , where  $\varphi(x) = e^{-x}$ , from Example 4.2 both have a simple purely absolutely continuous spectrum filling in the interval  $[0, 1]$ . Therefore,  $D(\lambda)$  is the zero operator for all  $\lambda \in \mathbb{R} \setminus (0, 1)$ .

(\*) The function  $\varphi$  is cyclic for  $T$ .

Hence, Theorem 4.3 implies that the kernel of  $D(\lambda)$  is trivial for all  $0 < \lambda < 1$ .

Furthermore, an application of Proposition 6.1 yields that  $D(\lambda)$  is unitarily equivalent to a bounded self-adjoint Hankel operator for all  $\lambda$  in  $\mathbb{R}$  except for at most countably many  $\lambda$  in  $[0, 1]$ .

Note that, in this example, explicit computations show that there are no exceptional points (see [16]).

*Proof of (\*).* Let  $k$  be in  $\mathbb{N}_0$ . Define the  $k$ th Laguerre polynomial  $L_k$  on  $(0, \infty)$  by  $L_k(x) := \frac{e^x}{k!} \frac{d^k}{dx^k}(x^k e^{-x})$ . Furthermore, define  $\psi_k$  on  $(0, \infty)$  by  $\psi_k(x) := x^k e^{-x}$ . A straightforward computation shows that

$$(A_0 \psi_k)(x) = \frac{1}{2} e^{-x} \left\{ \frac{x^{k+1}}{k+1} + \frac{1}{2^{k+1}} \sum_{\ell=0}^{k-1} (2x)^{k-\ell} \frac{k!}{(k-\ell)!} \right\}.$$

By induction on  $n \in \mathbb{N}_0$ , it easily follows that  $p \cdot \varphi$  belongs to the linear span of  $A_0^\ell \varphi$ ,  $\ell \in \mathbb{N}_0$ ,  $\ell \leq n$ , for all polynomials  $p$  of degree  $\leq n$ .

In particular, the functions  $\phi_j$  defined on  $(0, \infty)$  by  $\phi_j(x) := \sqrt{2} L_j(2x) e^{-x}$  are elements of  $\text{span}\{A_0^\ell \varphi : \ell \in \mathbb{N}_0, \ell \leq n\}$  for all  $j \in \mathbb{N}_0$  with  $j \leq n$ .

Since  $(\phi_j)_{j \in \mathbb{N}_0}$  is an orthonormal basis of  $L^2(0, \infty)$ , it follows that  $\varphi$  is cyclic for  $T$ .  $\square$

Example 7.2 suggests the conjecture that Proposition 6.1 (2) can be strengthened to hold up to a finite exceptional set.

Last, we consider different examples where the multiplicity in the spectrum of  $T$  is such that we can apply Proposition 4.1.

**Example 7.3.** (1) Let  $T$  be an arbitrary orthogonal projection on  $\mathfrak{H}$ , and let  $S$  be a self-adjoint operator of finite rank. Then zero or one is an eigenvalue of  $T$  with infinite multiplicity, and we can apply Proposition 4.1.

(2) Put  $\mathfrak{H} = L^2(0, \infty)$  and let  $T$  be the Carleman operator, i. e., the bounded Hankel operator such that

$$(Tg)(x) = \int_0^\infty \frac{g(y)}{x+y} dy$$

for all continuous functions  $g : (0, \infty) \rightarrow \mathbb{C}$  with compact support.

*It is well known (see, e. g., [26, Chapter 10, Theorem 2.3]) that the Carleman operator has a purely absolutely continuous spectrum of uniform multiplicity two filling in the interval  $[0, \pi]$ . Therefore, if  $S$  is any self-adjoint operator of rank one, Proposition 4.1 can be applied.*

**Jacobi operators.** Consider a bounded self-adjoint Jacobi operator  $H$  acting on the Hilbert space  $\ell^2(\mathbb{Z})$  of complex square summable two-sided sequences. More precisely, suppose that there exist bounded real-valued sequences  $a = (a_n)_n$  and  $b = (b_n)_n$  with  $a_n > 0$  for all  $n \in \mathbb{Z}$  such that

$$(Hx)_n = a_n x_{n+1} + a_{n-1} x_{n-1} + b_n x_n, \quad n \in \mathbb{Z},$$

cf. [36, Theorem 1.5 and Lemma 1.6]. The following result is well known.

**Proposition 7.4** (see [36], Lemma 3.6). *Let  $H$  be a bounded self-adjoint Jacobi operator on  $\ell^2(\mathbb{Z})$ . Then the singular spectrum of  $H$  has spectral multiplicity one, and the absolutely continuous spectrum of  $H$  has multiplicity at most two.*

In the case where  $H$  has a simple spectrum, there exists a cyclic vector  $\varphi$  for  $H$ , and we can apply Proposition 6.1 to  $H$  with the rank one perturbation  $S = \langle \cdot, \varphi \rangle \varphi$ .

Otherwise,  $H$  fulfills condition (3) with  $N = 1$  in Proposition 4.1. Let us discuss some examples in the latter case with  $T = H$ . Since  $S$  can be an arbitrary self-adjoint operator of rank one, we do not mention it in the following.

**Example 7.5.** *Consider the discrete Schrödinger operator  $H = H_V$  on  $\ell^2(\mathbb{Z})$  with bounded potential  $V : \mathbb{Z} \rightarrow \mathbb{R}$ ,*

$$(H_V x)_n = x_{n+1} + x_{n-1} + V_n x_n, \quad n \in \mathbb{Z}.$$

*If the spectrum of  $H_V$  contains only finitely many points outside of the interval  $[-2, 2]$ , then [6, Theorem 2] implies that  $H_V$  has a purely absolutely continuous spectrum of multiplicity two on  $[-2, 2]$ .*

*It is well known that the free Jacobi operator  $H_0$  with  $V = 0$  has a purely absolutely continuous spectrum of multiplicity two filling in the interval  $[-2, 2]$ .*

Let us consider the almost Mathieu operator  $H = H_{\kappa, \beta, \theta} : \ell^2(\mathbb{Z}) \rightarrow \ell^2(\mathbb{Z})$  defined by

$$(Hx)_n = x_{n+1} + x_{n-1} + 2\kappa \cos(2\pi(\theta + n\beta))x_n, \quad n \in \mathbb{Z},$$

where  $\kappa \in \mathbb{R} \setminus \{0\}$  and  $\beta, \theta \in \mathbb{R}$ . In fact, it suffices to consider  $\beta, \theta \in \mathbb{R}/\mathbb{Z}$ .

The almost Mathieu operator plays an important role in physics, see, for instance, the review [18] and the references therein.

Here, we are interested in cases where Proposition 4.1 can be applied to the almost Mathieu operator with an arbitrary self-adjoint rank one perturbation. Sufficient conditions for this purpose are provided in the following lemma.

**Lemma 7.6.** *(1) If  $\beta$  is rational, then for all  $\kappa$  and  $\theta$  the almost Mathieu operator  $H_{\kappa, \beta, \theta}$  is periodic and has a purely absolutely continuous spectrum of uniform multiplicity two.*

*(2) If  $\beta$  is irrational and  $|\kappa| < 1$ , then for all  $\theta$  the almost Mathieu operator  $H_{\kappa, \beta, \theta}$  has a purely absolutely continuous spectrum of uniform multiplicity two.*

*Proof.* (1) If  $\beta$  is rational, then  $H_{\kappa, \beta, \theta}$  is a periodic Jacobi operator. Hence, it is well known (see, e. g., [36, p. 122]) that the spectrum of  $H_{\kappa, \beta, \theta}$  is purely absolutely



continuous. According to [8, Theorem 9.1], we know that the absolutely continuous spectrum of  $H_{\kappa,\beta,\theta}$  is uniformly of multiplicity two. This proves (1).

(2) Suppose that  $\beta$  is irrational. Avila has shown (see [2, Main Theorem]) that the almost Mathieu operator  $H_{\kappa,\beta,\theta}$  has a purely absolutely continuous spectrum if and only if  $|\kappa| < 1$ . Again, [8, Theorem 9.1] implies that the absolutely continuous spectrum of  $H_{\kappa,\beta,\theta}$  is uniformly of multiplicity two. This finishes the proof.  $\square$

Problems 4–6 of Simon’s list [34] are concerned with the almost Mathieu operator. Avila’s result [2, Main Theorem], which we used in the above proof, is a solution for Problem 6 in [34].

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#### REFERENCES

- [1] J. Arazy, L. Zelenko, *Finite-dimensional perturbations of self-adjoint operators*, Integral Equations Operator Theory **34** (1999), 127–164
- [2] A. Avila, *The absolutely continuous spectrum of the almost Mathieu operator*, e-print arXiv:0810.2965v1 [math.DS] (2008)
- [3] H. Behncke, *Finite dimensional perturbations*, Proc. Amer. Math. Soc. **72** (1978), 82–84
- [4] M. Sh. Birman, M. Solomyak, *Double operator integrals in a Hilbert space*, Integral Equations Operator Theory **47** (2003), 131–168
- [5] M. S. Birman, M. Z. Solomjak, *Spectral Theory of Self-Adjoint Operators in Hilbert Space*, D. Reidel Publishing Co., Dordrecht, 1987
- [6] D. Damanik, R. Killip, B. Simon, *Schrödinger operators with few bound states*, Comm. Math. Phys. **258** (2005), 741–750
- [7] C. Davis, *Separation of two linear subspaces*, Acta Sci. Math. Szeged **19** (1958), 172–187
- [8] P. Deift, B. Simon, *Almost periodic Schrödinger operators. III. The absolutely continuous spectrum in one dimension*, Comm. Math. Phys. **90** (1983), 389–411
- [9] Ju. B. Farforovskaja, *An example of a Lipschitzian function of selfadjoint operators that yields a nonnuclear increase under a nuclear perturbation*, Zap. Naučn. Sem. Leningrad. Otdel. Mat. Inst. Steklov. (LOMI) **30** (1972), 146–153 (Russian)
- [10] P. R. Halmos, *Two subspaces*, Trans. Amer. Math. Soc. **144** (1969), 381–389
- [11] J. S. Howland, *Spectral theory of self-adjoint Hankel matrices*, Michigan Math. J. **33** (1986), 145–153
- [12] V. Jakšić, Y. Last, *A new proof of Poltoratskii’s theorem*, J. Funct. Anal. **215** (2004), 103–110
- [13] A. Klenke, *Probability Theory*, Springer, London, 2014
- [14] A. W. Knap, *Basic Real Analysis*, Birkhäuser, Boston, 2005
- [15] V. Kostrykin, K. A. Makarov, *On Krein’s example*, Proc. Amer. Math. Soc. **136** (2008), 2067–2071
- [16] M. G. Kreĭn, *On the trace formula in perturbation theory*, Mat. Sbornik N.S. **33(75)** (1953), 597–626 (Russian)
- [17] M. G. Kreĭn, *Some new studies in the theory of perturbations of self-adjoint operators*, in *First Math. Summer School, Part I*, Naukova Dumka, Kiev, 1964, pp. 103–187 (Russian). English transl.: *On certain new studies in the perturbation theory for selfadjoint operators*, in *Topics in Differential and Integral Equations and Operator Theory*, Birkhäuser, Basel, 1983, pp. 107–172
- [18] Y. Last, *Spectral theory of Sturm-Liouville operators on infinite intervals: a review of recent developments*, in *Sturm-Liouville Theory*, Birkhäuser, Basel, 2005, pp. 99–120



- [19] C. Liaw, S. Treil, *Rank one perturbations and singular integral operators*, J. Funct. Anal. **257** (2009), 1947–1975
- [20] R. A. Martínez-Avendaño, *Essentially Hankel operators*, J. London Math. Soc. (2) **66** (2002), 741–752
- [21] R. A. Martínez-Avendaño, S. R. Treil, *An inverse spectral problem for Hankel operators*, J. Operator Theory **48** (2002), 83–93
- [22] A. V. Megretskii, V. V. Peller, S. R. Treil, *The inverse spectral problem for self-adjoint Hankel operators*, Acta Math. **174** (1995), 241–309
- [23] K. R. Parthasarathy, *Introduction to Probability and Measure*, Hindustan Book Agency, New Delhi, 2005
- [24] V. V. Peller, *Hankel operators in the perturbation theory of unitary and self-adjoint operators*, Funktsional. Anal. i Prilozhen. **19** (1985), 37–51 (Russian).  
English transl.: Functional Anal. Appl. **19** (1985), 111–123
- [25] V. V. Peller, *Hankel operators in the perturbation theory of unbounded self-adjoint operators*, in *Analysis and Partial Differential Equations*, Lecture Notes in Pure and Appl. Math. **122**, Dekker, New York, 1990, pp. 529–544
- [26] V. V. Peller, *Hankel Operators and Their Applications*, Springer, New York, 2003
- [27] A. Pushnitski, *The scattering matrix and the differences of spectral projections*, Bull. Lond. Math. Soc. **40** (2008), 227–238
- [28] A. Pushnitski, *Spectral theory of discontinuous functions of self-adjoint operators: essential spectrum*, Integral Equations Operator Theory **68** (2010), 75–99
- [29] A. Pushnitski, *Scattering matrix and functions of self-adjoint operators*, J. Spectr. Theory **1** (2011), 221–236
- [30] A. Pushnitski, D. Yafaev, *Spectral theory of discontinuous functions of self-adjoint operators and scattering theory*, J. Funct. Anal. **259** (2010), 1950–1973
- [31] M. Rosenblum, *On the Hilbert matrix, I*, Proc. Amer. Math. Soc. **9** (1958), 137–140
- [32] M. Rosenblum, *On the Hilbert matrix, II*, Proc. Amer. Math. Soc. **9** (1958), 581–585
- [33] K. Schmüdgen, *Unbounded Self-adjoint Operators on Hilbert Space*, Springer, Dordrecht, 2012
- [34] B. Simon, *Schrödinger operators in the twenty-first century*, in *Mathematical Physics 2000*, Imperial College Press, London, 2000, pp. 283–288
- [35] B. Simon, *Schrödinger operators with purely discrete spectrum*, Methods Funct. Anal. Topology **15** (2009), 61–66
- [36] G. Teschl, *Jacobi Operators and Completely Integrable Nonlinear Lattices*, American Mathematical Society, Providence, RI, 2000
- [37] F.-Y. Wang, J.-L. Wu, *Compactness of Schrödinger semigroups with unbounded below potentials*, Bull. Sci. Math. **132** (2008), 679–689

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